Cheap Sine with $< 10^{-4}$ Error

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ABSTRACT. We are looking for a computationally cheap approximating polynomial for the sine and cosine functions. We start from the TAYLOR expansion, restricing us to three terms, computation being done by $x\,[1+x^2\,(B+C\,x^2)]$ or $x\,[A+x^2\,(B+C\,x^2)]$. We want to stick to $\sin 0=0$ and $\sin\frac{\pi}{2}=1$. Furthermore we do not want to need a division, like BHASKARA's formula or using PADÉ approximants do.

1. Starting from Taylor

The series expansion for $\sin x$ is

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2\,n+1)!},$$

and so the first few terms are

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots$$

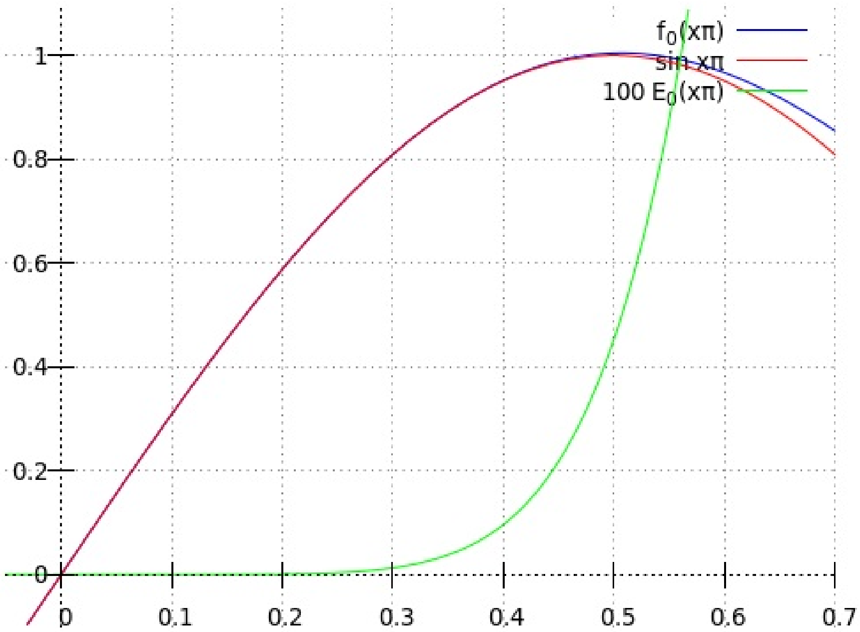
What can we achieve restricting ourselves to 5th order (3 terms)?

$$f_0(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}=x\left(1-x^2\left(\frac{1}{6}-\frac{1}{120}\,x^2\right)\right) \qquad (1)$$

$f_0(x)$ can be computed by a program with four multiplies incl. the one to get x^2. Our error will be

$$E_0(x)=f_0(x)-\sin x=\frac{x^7}{7!}-\frac{x^9}{9!}+\frac{x^{11}}{11!}-\cdots+\cdots$$

which is monotonically growing inside the interval $0\le x\le\frac{\pi}{2}$ - see the plot beneath. So the maximal



error is $E_0(\frac{\Pi}{2}) = 0.0045249$ which amounts to 0.45 percent. That's not too bad, but the error is concentrated near $\frac{\Pi}{2}$. In addition, feeding a value >1.0 to an arcsin or arccos function might be harming. Can we improve without entering higher orders? Let's look at some basic propeties of the sine function;

$$\sin 0 = 0, \quad \sin\frac{\Pi}{2} = 1, \tag{2}$$
$$\sin' 0 = \cos 0 = 1, \quad \sin'\frac{\Pi}{2} = \cos\frac{\Pi}{2} = 0.$$

And let's look at a function

$$f(x) = x - a\,x^3 + b\,x^5 = x\,(1 - x^2\,(a - b\,x^2)), \tag{3}$$

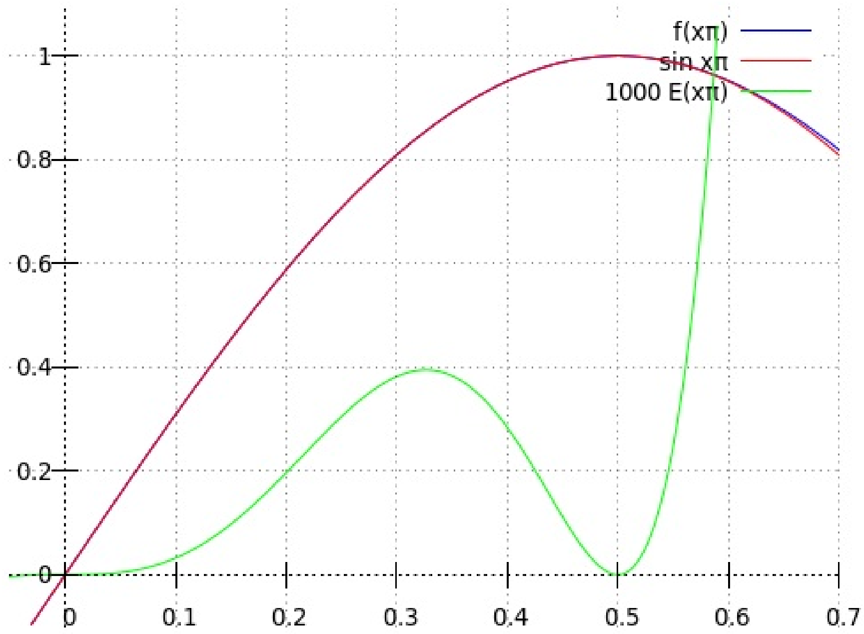
$$f'(x) = 1 - 3\,a\,x^2 + 5\,b\,x^4.$$

$f(0)=0$ and $f'(0) = 1$ we get "for free", irrespective of the values for a and b. Then let's choose a and b in such a way that

$$f(\frac{\Pi}{2}) = 1, \quad f'(\frac{\Pi}{2}) = 0.$$

We get

$$a = \frac{4}{\pi^3}(2\,\pi - 5), \quad b = \frac{16}{\pi^5}(\pi - 3)\,.$$



As the plot shows, we end up with a maximal error of about 0.04 %, 1/10 of what we got from $f_0(x)$. Explicitly we find the error maximim at

$$\begin{aligned} x_{\text{maxerr}} &= 0.32639327832147\,13\,\pi, \\ E_{\text{maxerr}} = f(x_{\text{maxerr}}) - \sin x_{\text{maxerr}} &= 3.945343147131463 \times 10^{-4}. \end{aligned}$$

What about arguments outside the interval $0 \le x \le \frac{\pi}{2}$? As is $\sin x$, our approximating $f(x)$ is an odd function: $f(-x) = -f(x)$. So we got covered x range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Furthermore we have

$$\sin(x + \pi) = -\sin x \quad , \quad \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

which, together with the fundamental 2π periodicity, gets us fully covered for cosine as well as sine throughout the argument range $-\infty \le x \le \infty$.

Can we do better by spreading the error more evenly in our interval? Let's look at a slightly different approximating function, keeping the three term restriction

$$g(x) = A\,x + B\,x^3 + C\,x^5 \tag{4}$$

$$g'(x) = A + 3\,B\,x^2 + 5\,C\,x^4$$

If we want to keep all the properties of the sine function given in (2), then $g'(0) = 1$, so $A = 1$. With this for the rest we are back at the determination of the two constants in (3). So we will have to let go some of our requirements, maybe best those concernig $g'(x)$. We will stick to $A = 1$; so we keep the same complexity as with $f(x)$. Hence we are now looking for optimal B and C in

$$g(x) = x + B\,x^3 + C\,x^5 = x\,(1 + x^2\,(B + C\,x^2)).$$

The condition $g'\left(\frac{\pi}{2}\right) = 0$ we will replace by the insertion of a third sample point x_1 between zero and $\frac{\pi}{2}$ at which $g(x_1)$ hits $\sin x_1$. In summary

$$\begin{array}{rcl} g(x_1) & = & \sin x_1 \\ g\left(\dfrac{\pi}{2}\right) & = & 1 \end{array}$$

Coefficients B and C will now depend on our choice of x_1.

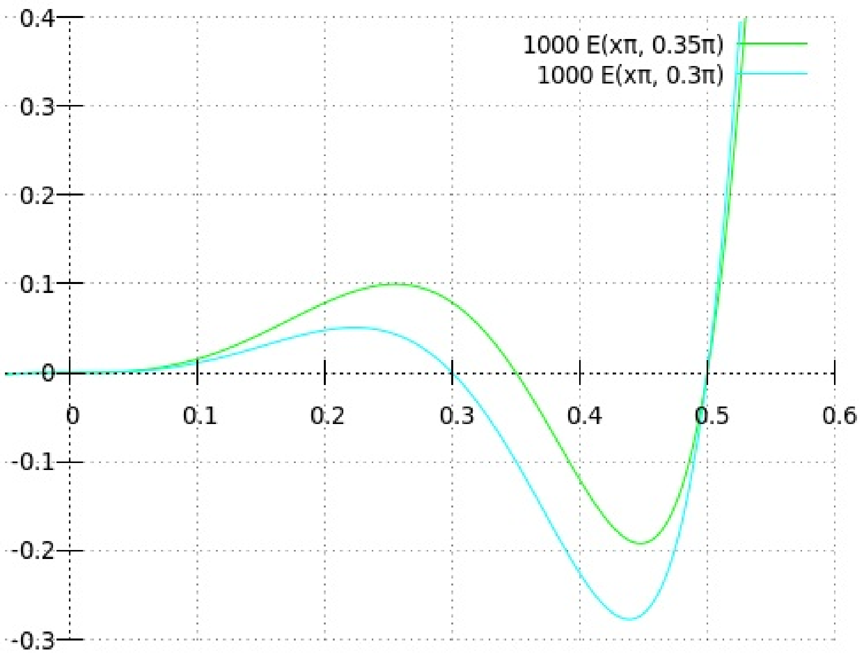
$$\begin{pmatrix} x_1^3 & x_1^5 \\ \left(\frac{\pi}{2}\right)^3 & \left(\frac{\pi}{2}\right)^5 \end{pmatrix} \cdot \begin{pmatrix} B(x_1) \\ C(x_1) \end{pmatrix} = \begin{pmatrix} \sin x_1 - x_1 \\ 1 - \frac{\pi}{2} \end{pmatrix}$$

$$\begin{pmatrix} B(x_1) \\ C(x_1) \end{pmatrix} = \frac{1}{4\,x_1^2 - \pi^2} \begin{pmatrix} -\frac{\pi^2}{x_1^3} & \frac{32\,x_1^2}{\pi^3} \\ \frac{4}{x_1^3} & -\frac{32}{\pi^3} \end{pmatrix} \cdot \begin{pmatrix} \sin x_1 - x_1 \\ 1 - \frac{\pi}{2} \end{pmatrix} \tag{5}$$

Our error function now is

$$E(x, x_1) = g(x, x_1) - \sin x = x + B(x_1)\,x^3 + C(x_1)\,x^5 - \sin x.$$

Here is a plot of it for two different x_1.



The error function's derivative is

$$\frac{d}{\mathrm{dx}}E(x,x_1) = 1-\cos(x) + 5\,C(x_1)\,x^4 + 3\,B(x_1)\,x^2$$

Hence the extrema of $E(x,x_1)$ along the positive axis are where

$$\frac{d}{\mathrm{dx}}E(x,x_1) = 1-\cos(x) + 3\,B(x_1)\,x^2 + 5\,C(x_1)\,x^4 = 0, \tag{6}$$

Let these roots of (6) be at x_a and at x_b - we can only find them numerically, but both are uniquely determined by the choice of x_1, so we can write $x_a(x_1)$ and $x_b(x_1)$. Then the two extrema themselves have error values

$$\begin{aligned} E(x_a,x_1) &= x_a - \sin x_a + B(x_1)\,x_a^3 + C(x_1)\,x_a^5, \\ E(x_b,x_1) &= x_b - \sin x_b + B(x_1)\,x_b^3 + C(x_1)\,x_b^5. \end{aligned} \tag{7}$$

Following CHEBYSHEV's *equioscillation theorem*[1] and because one must be a maximum and the other one a minimum, the optimal x_1 will be where $E(x_a,x_1) + E(x_b,x_1) = 0$. Thus we have to solve

$$x_a(x_1) + x_b(x_1) + B(x_1)\,(x_a(x_1)^3 + x_b(x_1)^3) + C(x_1)\,(x_a(x_1)^5 + x_b(x_1)^5) = \sin x_a(x_1) + \sin x_b(x_1) \tag{8}$$

for the optimal x_1. As already (6) has to be solved numerically, this refers to (8) as well. A small C program has been written to do this job. Roughly it did

With a given x_1;
    Compute B and C from (5);
    Find x_a and x_b as the roots of (6);
    Evaluate the error at x_a and x_b from (7);
Repeat the sequence above for some range of x_1, tracking the sum of errors to find it's minimum.
    Start with a broad range and big steps for x_1, then refine.

The results are as follows. Error sum = 0.000000000000e+00 for

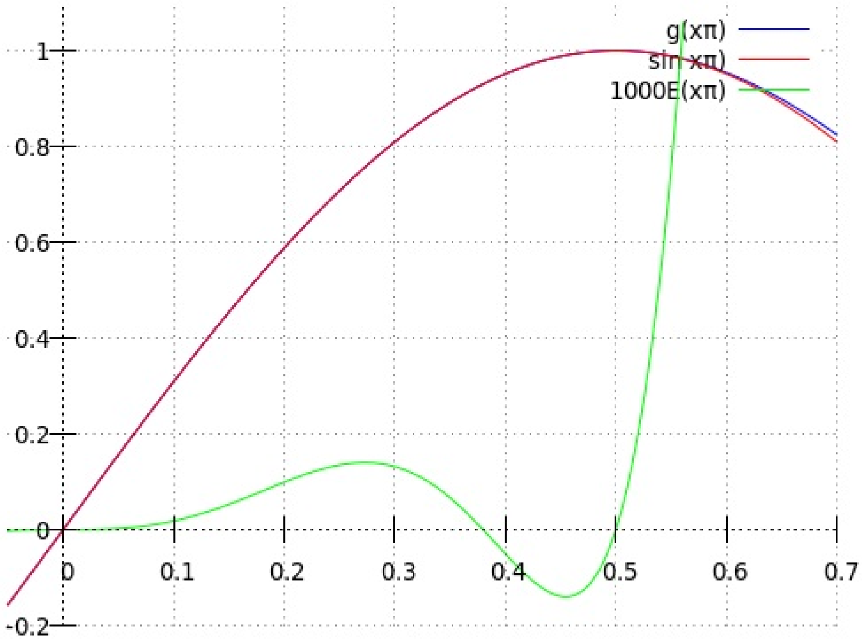
$$\begin{aligned} x_1 &= 1.193500035027 = 0.379902860310\,\pi, & & \\ x_b &= 0.857212658775 = 0.272859263850\,\pi, & E(x_b,x_1) &= 0.000140012094 \\ x_a &= 1.428536909037 = 0.454717420925\,\pi, & E(x_a,x_1) &= -0.000140012094 \\ B(x_1) &= -1.660059992381e-01, & C(x_1) &= 7.592417840901e-03. \end{aligned}$$

Thus in summary our approximating function gets

$$\begin{aligned} g(x) &= x - 1.660059992381 \times 10^{-1}\,x^3 + 7.592417840901 \times 10^{-3}\,x^5 \\ &= x\,[1 - x^2\,(1.660059992381 \times 10^{-1} + 7.592417840901 \times 10^{-3}\,x^2)] \end{aligned}$$

and we end up with a maximal error of $\pm 0.14\,{}^0/_{00}$. We see the distribution of the error in the plot below.

1. See http://www.math.uiowa.edu/~jeichhol/qual prep/Notes/cheb-equiosc-thm_2007.pdf, or https://www.maa.org/sites/default/files/images/upload_library/4/vol6/Mayans/Contents.html.



Requesting $g'(0)=1$ (as $\sin' 0=1$) has automatically led to $E'(0)=0$. So the error function starts very softly from $x=0$. Can we achieve smaller absolute extrema by allowing the error to rise more sharply? This means dropping $g'(0)=1$, so we are back to the full $g(x)$ from equation (4), and to this list of requirements:

$$\begin{aligned} g(0) &= 0, \\ g\left(\frac{\pi}{2}\right) &= 1, \\ g(x_1) &= \sin x_1, \\ E'(x_a)=0, &\quad E'(x_b)=0, \\ E(x_a)+E(x_b) &= 0. \end{aligned}$$

This will now be attacked - with a little bit more help by Pafnuty L. Chebyshev.

2. Chebyshev

Since we have to omit $A=0$ and therefore have to find three coefficients, we may profitably omit the single interpolation node x_1 inside the zero to $\pi/2$ interval, and use two nodes x_1 and x_2 instead. This gives us a system of three equations

$$\begin{aligned} g(x_1) &= \sin x_1, \\ g(x_2) &= \sin x_2, \\ g\left(\frac{\pi}{2}\right) &= 1. \end{aligned} \tag{9}$$

How to know the optimal x_1 and x_2? To search like above "by brute force" seems impractible. Here theory comes in and tells us that the *Chebyshev nodes*[2] may at least be a good start. These nodes are given by the appropriately scaled zeros of a *Chebyshev polynomial* (of the first kind). The one of these that has four zeros inside the closed interval [0,1] is $T_7(x)$ - it has seven zeros in [-1,1].

$$T_7(x)=64\,x^7-112\,x^5+56\,x^3-7\,x,$$

and it is obeying the functional equation

$$T_7(\cos\xi)=\cos(7\,\xi).$$

2. See https://en.wikipedia.org/wiki/Chebyshev_nodes.

From this we know the zeroes and that ones in [0,1] are at

$$\cos\frac{m\,\pi}{14} \quad , \quad m = 7, 5, 3, 1.$$

We have to scale $T_7(x)$ in such a way that the zeros get mapped to 0, x_1, x_2, and $\frac{\pi}{2}$. That is, we have to stretch x by a factor $\frac{\pi}{2\cos\frac{\pi}{14}}$, and we get

$$\begin{aligned} x_1 &= \frac{\cos\frac{3\pi}{14}}{\cos\frac{\pi}{14}}\frac{\pi}{2} = \left(2\cos^2\left(\frac{\pi}{14}\right) - \frac{3}{2}\right)\pi, \\ x_2 &= \frac{\cos\frac{5\pi}{14}}{\cos\frac{\pi}{14}}\frac{\pi}{2} = \left(8\cos^4\left(\frac{\pi}{14}\right) - 10\cos^2\left(\frac{\pi}{14}\right) + \frac{5}{2}\right)\pi. \end{aligned}$$

Given the interpolation nodes, lets solve equations (9).

$$\begin{bmatrix} x_1 & x_1{}^3 & x_1{}^5 \\ x_2 & x_2{}^3 & x_2{}^5 \\ \frac{\pi}{2} & \left(\frac{\pi}{2}\right)^3 & \left(\frac{\pi}{2}\right)^5 \end{bmatrix} . \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sin x_1 \\ \sin x_2 \\ 1 \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \frac{1}{D}\begin{pmatrix} -\frac{\pi^3 x_2^3 (4x_2^2 - \pi^2)}{32} & \frac{\pi^3 x_1^3 (4x_1^2 - \pi^2)}{32} & x_1^3 x_2^3 (x_2^2 - x_1^2) \\ \frac{\pi x_2 (16 x_2^4 - \pi^4)}{32} & -\frac{\pi x_1 (16 x_1^4 - \pi^4)}{32} & -x_1 x_2 (x_2^4 - x_1^4) \\ -\frac{\pi x_2 (4x_2^2 - \pi^2)}{8} & \frac{\pi x_1 (4x_1^2 - \pi^2)}{8} & x_1 x_2 (x_2^2 - x_1^2) \end{pmatrix} . \begin{bmatrix} \sin x_1 \\ \sin x_2 \\ 1 \end{bmatrix},$$

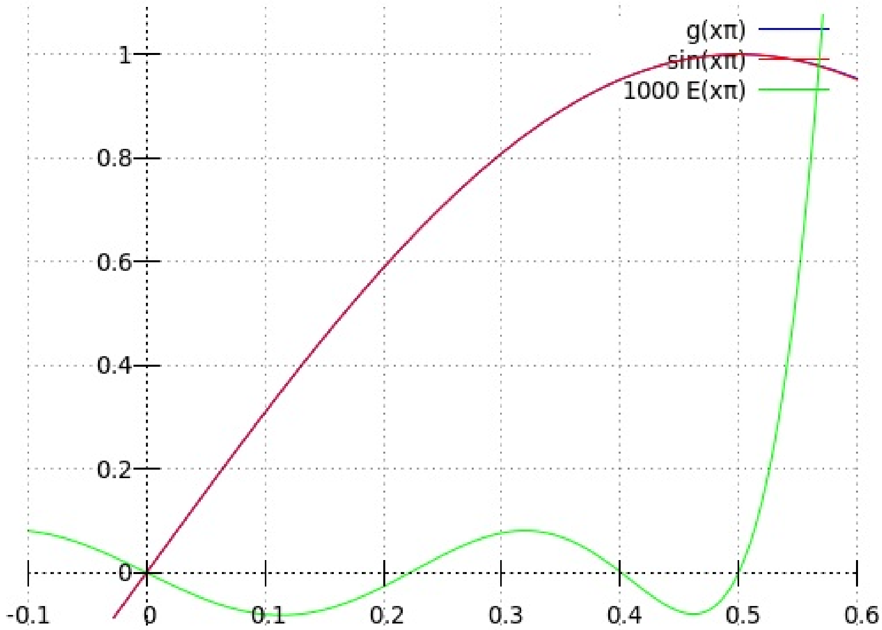
where D is the determinant of the matrix on the left side of equation (10). Numerical evaluation yields the result

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0.9996436199979504 \\ -0.1655633385816708 \\ 0.007471551686678179 \end{bmatrix},$$

and so we have from equation (4)

$$\begin{aligned} g(x) &= 0.9996436199979504\,x + -0.1655633385816708\,x^3 + 0.007471551686678179\,x^5, \\ &= x\,(0.9996436199979504 + x^2\,(0.007471551686678179\,x^2 - 0.1655633385816708)). \end{aligned}$$

Here $g(x)$ is plotted together with it's error function $E(x)$.



The extrema of $E(x)$ are given by

$$\begin{aligned} x_a &= 0.1139588721403732\,\pi, & E(x_a) &= -8.187880151318966\times 10^{-5},\\ x_b &= 0.3194692485308592\,\pi, & E(x_b) &= 8.08648074913515\times 10^{-5},\\ x_c &= 0.4619395981114392\,\pi, & E(x_c) &= -7.962173707186382\times 10^{-5}. \end{aligned}$$

As we see, we do not have perfect equioscillation. However the differences seem bearably small. To approach equioscillation more perfectly, techniques of the REMEZ algorithm[3] may be used for shifting x_a, x_b, x_c iteratively to their optimal positions. Here this is left to the inclined reader.

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3. See https://en.wikipedia.org/wiki/Remez_algorithm.